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HAMILTONIAN DYNAMICS AND STABILITY ANALYSIS OF NEUTRAL ELECTROMAGNETIC FLUIDS WITH INDUCTION

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The Lie–Poisson Hamiltonian structure of the special-relativistic electromagnetic fluid equations is derived. This Hamiltonian structure provides synthesis and insight leading to new conservation laws and stability conditions for the equilibrium solutions. A corollary of the stability results generalizes Rayleigh's inflectional instability criterion for ideal incompressible fluids to the present case. Another alternative, Hamiltonian formulation of relativistic electromagnetic fluid dynamics is constructed systematically via Lie-algebraic considerations of the Poisson bracket. (In particular, relativistic magnetohydrodynamics emerges naturally from these considerations.) The nonrelativistic limits of these two formulations are also determined and are shown to be regular and to preserve the corresponding Lie–Poisson structures.

1. Introduction

Electromagnetic fluids (EMF) include inviscid, compressible, perfectly conducting, polarizable and magnetizable fluids, with generally nonlinear polarization and magnetization properties. Penfield and Haus [1, 2] derive the special relativistic equations and determine the total stress-energy tensor for EMF from a constrained variational principle. The Penfield–Haus variational principle leads to a covariant "Clebsch representation" of the EMF 4-momentum density (see also eq. (3.1)) of the general form

$$M_\mu = -\sum_I P_{(I)} q^{(I)}_{,\mu}, \quad \mu = 0,1,2,3; \quad I \in \{\text{canonical coordinates}\}, \tag{1.1}$$

where $q^{(I)}_{,\mu} = \partial_\mu q^{(I)} = \partial q^{(I)}/\partial x^\mu$ denotes the partial derivative, and $P_{(I)}$ and $q^{(I)}$ in (1.1) are canonically conjugate fields obtained by passing via the usual Legendre transformations from the Penfield–Haus variational principle to the $3+1$ Hamiltonian formulation of the EMF equations, expressed as a dynamical system in the laboratory Lorentz frame.

Not all of the canonical Clebsch variables $(P_{(I)}, q^{(I)})$ in (1.1) are physically significant. Indeed, the relation of these canonical variables to the physical momentum density is not even unique, due to the gauge freedom in (1.1). See Henyey [3] for a discussion of this Clebsch gauge freedom. The nonuniqueness can be eliminated and the connection to proper physical variables can be made as done in Holm and Kupershmidt [4] for nonrelativistic continuum theories by using the Clebsch relation (1.1) to map the $3+1$ Hamiltonian formulation in the laboratory frame from the larger space of canonically conjugate Clebsch variables to the smaller space of physical fluid variables. This results in a Poisson bracket that is not canonical in form, but is expressed solely in physical terms. As we shall discuss later (in section 5), the

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partition of the momentum density in (1.1) between matter and electromagnetic field is not uniquely defined for EMF. (This is the old Abraham–Minkowskii controversy.) This ambiguity will lead to physically equivalent, alternative, Hamiltonian formulations of EMF dynamics interrelated by Poisson maps, i.e., maps that preserve the values of Poisson brackets. In particular, relativistic magnetohydrodynamics will emerge naturally from EMF dynamics via such considerations of Poisson maps.

The "Clebsch map" based on (1.1) also preserves the values of the Poisson brackets among the physical variables and is analogous to the map for the rigid body in classical mechanics, from the six-dimensional space of the Euler angles and their canonically conjugate momenta to the three components of angular momentum in the body satisfying noncanonical Poisson bracket relations associated to the dual of the Lie algebra su(2). See, e.g., Sudarshan and Mukunda [5] and Marsden et al. [6].

The present work constructs the noncanonical $3+1$ Hamiltonian formulation for special relativistic EMF dynamics in the laboratory frame by using the Clebsch map derived from (1.1).

The noncanonical Poisson bracket for EMF dynamics appearing via the Clebsch map (1.1) turns out to be associated naturally to the dual of a Lie algebra of semidirect product type. Such Lie algebras are present as a generic feature of ideal fluid dynamics. See, e.g., Holm and Kupershmidt [4] and various articles in Marsden [7]. For EMF, the first hint of this Lie algebraic structure appears in the differential-geometric formulation of the generalized Kelvin's Theorem, eq. (2.50), expressed in terms of Lie derivatives and differential forms.

The semidirect-product Lie algebra associated to EMF dynamics has a nontrivial kernel; so the corresponding Poisson bracket possesses Casimirs, i.e., functionals C_F containing an arbitrary function F and satisfying $\{C_F, G\} = 0$ for *every* functional G in the space of physical variables. Of course, the Casimirs are conserved, i.e.

$$\partial_t C_F = \{H, C_F\} = 0, \tag{1.2}$$

since they Poisson commute with every physical fluid variable in the laboratory frame, and the Hamiltonian H appearing in (1.2) depends only on these physical variables. As one could expect, conservation of the Casimirs in the space of physical variables corresponds via Noether's theorem to the symmetries of the Penfield–Haus constrained variational principle for EMF under Clebsch gauge transformations, i.e., canonical transformations leaving the Clebsch representation (1.1) invariant.

Note that the Casimir conservation laws are *robust*, in the sense that they persist even when the original Hamiltonian in (1.2) is perturbed or altered *arbitrarily*, in the space of physical variables.

Besides providing conservation laws and geometric or Lie algebraic insight, the Casimirs are conservation laws that help characterize relative equilibrium states of the EMF equations. Namely, the EMF equilibria obeying a certain Bernoulli-type relation [given in eq. (4.6)] are critical points of the sum of the Hamiltonian, H, and the Casimirs, C_F, the latter containing an arbitrary function F. This critical point property,

$$\delta(H + C_F) = 0, \tag{1.3}$$

associating the equilibrium states to particular Casimirs is useful in establishing Lyapunov stability criteria for the corresponding equilibria by the so-called energy-Casimir stability method. This stability method is a development of the well-known Lyapunov method that uses the Hamiltonian formalism and Casimirs to construct the desired Lyapunov functionals for certain classes of equilibria; in particular, for those equilibria whose Bernoulli relation is expressible in explicit functional form, as in eq. (4.6).

Holm et al. [8] provides examples of nonrelativistic stability analyses for various models of ideal compressible fluids and plasmas by this method, and Holm and Kupershmidt [9] presents the stability analysis of relativistic ideal plasma equilibria using the energy-Casimir method. Abarbanel and Holm [10] characterizes the entire set of fluid equilibria (not just the aforementioned Bernoulli equilibria) as critical states satisfying (1.3) under an extended class of variations, including variations of the Lagrangian coordinate functions (fluid particle labels). (See also Holm, Marsden, and Ratiu [11], Part II, for a discussion of this approach.)

Various nonlinear, energy-conserving, *approximations* of EMF dynamics can be found by examining the Hamiltonian structures that are "nested" Lie-algebraically within the Hamiltonian structure for EMF. In particular, the relativistic dynamics and Hamiltonian structure for EMF can be specialized to the case of relativistic magnetohydrodynamics (MHD), by setting the magnetization and polarization to zero and using an alternative Poisson bracket for EMF obtained from the first one by a canonical (i.e., Poisson bracket preserving) map called the "entangling map". These two canonically equivalent Hamiltonian formulations of special-relativistic EMF (SREMF) also contain the following nonrelativistic models: the corresponding nonrelativistic version of EMF (see Holm [12]); the equations for nonrelativistic polarized fluids (with vanishing magnetization but nonzero polarization, which is possible for EMF in the nonrelativistic limit, see Calkin [3]); and nonrelativistic MHD. All of these nonrelativistic fluid plasma models derive from the regular limit of SREMF when c^{-2} tends to zero, with c denoting the speed of light.

The general-relativistic *extensions* of EMF and MHD are also available in this context and have the same type of semidirect-product Lie–Poisson brackets, appearing in a direct sum with the canonical Poisson structure for the metric dynamics. The Hamiltonian structures for these general relativistic extensions can be constructed by applying either the methods of Ray [14] to modify the Penfield–Haus action principle, or those of Bao et al. [15] and Holm [16] to modify the present Hamiltonian formulation so as to include the dynamics of the space-like Riemannian metric and its canonically conjugate momentum field.

Plan. Section 2 gives the special-relativistic EMF equations and defines notation. In particular, the Penfield–Haus variational principle is recalled and the EMF equations are derived from it in standard 4-vector notation. Section 3 passes from the constrained variational principle of Penfield and Haus [1] to the Hamiltonian description of relativistic EMF dynamics in canonical Clebsch coordinates $(P_{(I)}, q^{(I)})$, by employing a $3+1$ spacetime split in the laboratory reference frame. Then we use the spatial part of the Clebsch map (1.1) in that reference frame to restrict the Hamiltonian description to physical fluid quantities. The resulting Poisson bracket in physical variables is shown to be associated to the dual of a certain semidirect-product Lie algebra. Dual coordinates and Casimirs for the Poisson bracket are identified in terms of physical variables. Section 4 discusses equilibrium conditions for the EMF equations. Equilibrium states are shown to correspond to critical points of the sum $H + C_F$, with H the Hamiltonian and C_F a Casimir containing an arbitrary function F. The Lyapunov stability of these equilibria is then investigated under the linearized dynamics. Section 5 derives several approximations of EMF dynamics in Hamiltonian form by considering special cases, canonical maps, and nonrelativistic limits of the Hamiltonian structure for EMF given in section 3.

This Hamiltonian approach with its resulting Lie-algebraic structure shows that ideal fluid dynamics – even relativistic fluid dynamics with electromagnetic induction – is essentially geometry in motion.

2. Fundamentals of special relativistic EMF

2.1. *Equations of motion and notation*

In Lorentz-covariant form, the special relativistic electromagnetic fluid (SREMF) equations are (Penfield and Haus [1], Lichnerowicz [17])

$$T^{\mu\nu}_{,\nu} = 0, \tag{2.1a}$$

$$(n'\bar{v}^{\nu})_{,\nu} = 0, \tag{2.1b}$$

$$\bar{v}^{\nu} s'_{,\nu} = 0, \tag{2.1c}$$

$$H^{\mu\nu}_{,\nu} = 0, \tag{2.2a}$$

$$\tilde{F}^{\mu\nu}_{,\nu} = 0. \tag{2.2b}$$

The SREMF equations (2.1) and (2.2) consist of local conservation laws for energy-momentum (2.1a) and number of particles (2.1b), the adiabatic condition (2.1c) for dissipationless flow, and the Maxwell equations (2.2a, b) including induction for a polarizable, magnetizable medium. In these equations, Greek indices run from 0 to 3, repeated indices are summed, and partial derivatives are denoted with a subscript comma (,). Superscript prime (′) denotes variables in the reference frame of a volume element moving with the fluid. For example, n' denotes the number of particles per unit volume and s' is the entropy per particle in the proper frame of the fluid (henceforth called the fluid frame). The quantity $\bar{v}^{\nu}$ denotes the timelike fluid velocity 4-vector, which becomes $\bar{v}^0 = c$, $\bar{v}^i = 0$, $i = 1,2,3$, in the fluid frame (Latin indices run from 1 to 3) and satisfies

$$g_{\mu\nu}\bar{v}^{\mu}\bar{v}^{\nu} = -c^2. \tag{2.3}$$

The metric tensor is given by the expression $-\mathrm{d}\tau^2 = g_{\mu\nu}\,\mathrm{d}x^{\mu}\,\mathrm{d}x^{\nu}$ for the proper time interval, $x^0 = ct$ being the real timelike coordinate. In the present work, we take $g_{\mu\nu}$ to be the Minkowskii metric, $\mathrm{diag}(-1,1,1,1)$ in the proper frame.

In the laboratory frame, $\bar{v}^{\mu}$ becomes $\bar{v}^0 = \gamma c$, $\bar{v}^i = \gamma v^i$, where v^i is the usual fluid velocity 3-vector and $\gamma^{-2} = 1 - v^2/c^2$, with $v^2 = v_i v^i$. Laboratory frame quantities will be unadorned and related to their primed fluid-frame counterparts by the standard Lorentz transformation rules (see, e.g., Møller [18]).

The quantity $T^{\mu\nu}$ in (2.1a) is the energy-momentum tensor, given in the fluid frame by (Penfield and Haus [1])

$$T'_{\mu\nu} = \begin{vmatrix} n'm_0c^2 + \varepsilon' & c^{-1}(\boldsymbol{E}'\times\boldsymbol{H}')_i \\ c^{-1}(\boldsymbol{E}'\times\boldsymbol{H}')_k & \begin{matrix}\delta_{ik}(\boldsymbol{E}'\cdot\boldsymbol{D}' + \boldsymbol{H}'\cdot\boldsymbol{B}' + P') \\ -D'_iE'_k - B'_iH'_k\end{matrix} \end{vmatrix}, \tag{2.4}$$

where $\boldsymbol{E}'$ is the electric field, $\boldsymbol{D}'$ is the displacement vector, $\boldsymbol{H}'$ is the magnetic field intensity, and $\boldsymbol{B}'$ is the magnetic induction, all in the fluid frame. The quantity from (2.4)

$$c^{-1}T'_{0i} = c^{-2}(\boldsymbol{E}'\times\boldsymbol{H}')_i \tag{2.5}$$

is the Poynting vector, which is the momentum density for EMF in the fluid frame. The zero-zero component of the energy-momentum tensor in (2.4)

$$T'_{00} = n'm_0c^2 + \varepsilon', \tag{2.6}$$

is the proper internal energy density including the rest mass contribution, with m_0 the particle rest mass. Note that ε' contains contributions from both the particles and the electromagnetic fields.

The energy density ε' in (2.6) satisfies the following thermodynamic identity in the fluid frame:

$$\mathrm{d}\varepsilon' = \boldsymbol{E}'\cdot\mathrm{d}\boldsymbol{D}' + \boldsymbol{H}'\cdot\mathrm{d}\boldsymbol{B}' + \left(\frac{P'+\varepsilon'}{n'}\right)\mathrm{d}n' + n'\theta'\,\mathrm{d}s', \tag{2.7}$$

where θ' is the proper temperature and P' is the total proper pressure, defined by

$$P' = n'\frac{\partial\varepsilon'}{\partial n'} - \varepsilon'. \tag{2.8}$$

Contraction of (2.1a) with $\bar{v}_\mu$ contributes the Poynting relation for power balance. In the fluid frame this relation is

$$\begin{aligned} 0 &= c^{-1}\bar{v}_\mu T^{\mu\nu}_{\ \ ,\nu} \\ [\text{by (2.4)}] &= \partial_\tau\left(n'm_0c^2 + \varepsilon'\right) + \mathrm{div}'\left(\boldsymbol{E}'\times\boldsymbol{H}'/c^2\right) \\ [\text{by (2.7)}] &= \boldsymbol{E}'\cdot\partial_\tau\boldsymbol{D}' + \boldsymbol{H}'\cdot\partial_\tau\boldsymbol{B}' + m_0c^2w\,\partial_\tau n' + n'\theta'\partial_\tau s' + \mathrm{div}'\left(\boldsymbol{E}'\times\boldsymbol{H}'/c^2\right), \end{aligned} \tag{2.9}$$

where w is the relativistic specific enthalpy,

$$w = 1 + \left(\frac{P'+\varepsilon'}{n'm_0c^2}\right), \tag{2.10}$$

∂_τ is the partial derivative with respect to proper time, τ, and div$'$ is the divergence in the fluid frame.

The representation of $T_{\mu\nu}$ in an arbitrary frame can now be found from (2.4) by using the standard Lorentz transformation laws (Møller [18], §66), see also (2.41) below.

The electromagnetic field tensor $F_{\mu\nu}$ is given by

$$c^{-1}F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}, \tag{2.11}$$

where A_μ is the 4-vector potential. The antisymmetric tensor $F_{\mu\nu}$ has electric and magnetic field components in the following form:

$$F_{\mu\nu} = \begin{vmatrix} 0 & E_1 & E_2 & E_3 \\ & 0 & -cB_3 & cB_2 \\ & & 0 & -cB_1 \\ & & & 0 \end{vmatrix}. \tag{2.12}$$

The dual tensor $\tilde{F}_{\mu\nu}$ in (2.2b) is obtained from $F_{\mu\nu}$ in (2.12) by replacing $\boldsymbol{E}$ by $-c\boldsymbol{B}$ and $c\boldsymbol{B}$ by $\boldsymbol{E}$,

$$\tilde{F}_{\mu\nu} = \begin{vmatrix} 0 & -cB_1 & -cB_2 & -cB_3 \\ & 0 & -E_3 & E_2 \\ & & 0 & -E_1 \\ & & & 0 \end{vmatrix}. \tag{2.13}$$

The field tensor in the medium $H^{\mu\nu}$ in (2.2a) is related formally to $F_{\mu\nu}$ by raising indices and replacing $\boldsymbol{E}$ by $\boldsymbol{D}$ and $\boldsymbol{B}$ by $\boldsymbol{H}/c^2$,

$$H^{\mu\nu} = \begin{vmatrix} 0 & -D_1 & -D_2 & -D_3 \\ & 0 & -H_3/c & H_2/c \\ & & 0 & -H_1/c \\ & & & 0 \end{vmatrix}. \tag{2.14}$$

The Maxwell equations resulting from (2.2a, b) in 3-vector language using (2.13) and (2.14) are

$$\operatorname{div} \boldsymbol{D} = 0, \tag{2.15a}$$

$$-\partial_t \boldsymbol{D} + \operatorname{curl} \boldsymbol{H} = 0, \tag{2.15b}$$

$$\operatorname{div} \boldsymbol{B} = 0, \tag{2.15c}$$

$$\partial_t \boldsymbol{B} + \operatorname{curl} \boldsymbol{E} = 0, \tag{2.15d}$$

which have the same form in any Lorentz frame. Note that (2.15a, c) are preserved in time, by the divergences of (2.15b, d), respectively. Thus, $\operatorname{div} \boldsymbol{D} = 0$ and $\operatorname{div} \boldsymbol{B} = 0$ can be taken as nondynamical constraints, i.e., can be assumed as initial conditions that are subsequently preserved.

2.2. *Penfield–Haus variational principle*

The relativistic EMF equations (2.1a–d) follow from Hamilton's principle,

$$\delta S = \delta \int \mathrm{d}t\, \mathrm{d}^3x\, L = 0, \tag{2.16}$$

where the constrained Lagrangian density L is a slight modification of that in Penfield and Haus [1] eq. (42),

$$L = L_0 - \xi \Gamma^{\mu}_{,\mu} - \beta \Gamma^{\mu} s'_{,\mu} - \lambda_{\Sigma} \Gamma^{\mu} X^{\Sigma}_{,\mu}, \tag{2.17}$$

with basic Lagrangian density

$$L_0 = \boldsymbol{E}' \cdot \boldsymbol{D}' - n' m_0 c^2 - \varepsilon'(n', s', \boldsymbol{D}', \boldsymbol{B}'), \tag{2.18}$$

and covariant particle flux density

$$\Gamma^{\mu} = n' \bar{v}^{\mu}. \tag{2.19}$$

In (2.17), the Lagrange multipliers ξ and β impose upon the extremals of S the EMF subsidiary equations (2.1b) and (2.1c), respectively; while λ_Σ imposes the "4-vector Lin constraint,"

$$\Gamma^\mu X^\Sigma_{,\mu} = 0, \quad \Sigma = 0,1,2,3, \tag{2.20}$$

on the trajectory of the fluid label X^Σ. As a consequence of the thermodynamic identity (2.7) and the definitions (2.8) and (2.10), the quantity L_0 in (2.18) satisfies the variational relation

$$\delta L_0 = \boldsymbol{D}' \cdot \delta \boldsymbol{E}' - \boldsymbol{H}' \cdot \delta \boldsymbol{B}' - n'\theta' \delta s' - m_0 c^2 w \, \delta n'. \tag{2.21}$$

For covariant equations to result, variational formulas such as expression (2.21) for the Lorentz scalar density δL_0 in the fluid frame need to be put into covariant form. The variations $\delta n'$, for example, depend on the velocity through the relation (2.3), or equivalently,

$$\bar{v}_\mu \bar{v}^\mu = -c^2. \tag{2.22}$$

The variation of $\Gamma^\mu = n'\bar{v}^\mu$ in (2.19) is

$$\delta \Gamma^\mu = n'\delta \bar{v}^\mu + \bar{v}^\mu \delta n'. \tag{2.23}$$

Since (2.22) implies the variational formula

$$\bar{v}_\mu \delta \bar{v}^\mu = \delta\left(\bar{v}_\mu \bar{v}^\mu / 2 \right) = 0, \tag{2.24}$$

we have, upon contracting (2.23) with $\bar{v}_\mu$ and using (2.22),

$$\delta n' = -\left(\bar{v}_\mu \delta \Gamma^\mu \right) / c^2. \tag{2.25}$$

Likewise, the variations $\delta \boldsymbol{E}'$ and $\delta \boldsymbol{B}'$ in (2.21) depend on the velocity via the Lorentz transformation formulas (Stratton [19], sec. 1.23),

$$\begin{aligned} \boldsymbol{E}' &= \boldsymbol{E}_\parallel + \gamma\left(\boldsymbol{E}_\perp + \boldsymbol{v} \times \boldsymbol{B} \right), \\ \boldsymbol{D}' &= \boldsymbol{D}_\parallel + \gamma\left(\boldsymbol{D}_\perp + \boldsymbol{v} \times \boldsymbol{H}/c^2 \right), \\ \boldsymbol{H}' &= \boldsymbol{H}_\parallel + \gamma\left(\boldsymbol{H}_\perp - \boldsymbol{v} \times \boldsymbol{D} \right), \\ \boldsymbol{B}' &= \boldsymbol{B}_\parallel + \gamma\left(\boldsymbol{B}_\perp - \boldsymbol{v} \times \boldsymbol{E}/c^2 \right), \end{aligned} \tag{2.26}$$

where $\parallel$ and $\perp$ subscripts refer to components parallel and perpendicular to $\boldsymbol{v}$, respectively. After a short calculation, eqs. (2.26) yield the variational relation (Penfield and Haus [1])

$$\boldsymbol{D}' \cdot \delta \boldsymbol{E}' - \boldsymbol{H}' \cdot \delta \boldsymbol{B}' = \tfrac{1}{2} H'_{\mu\nu} \delta F'^{\mu\nu} - \boldsymbol{R}' \cdot \delta \boldsymbol{v}, \tag{2.27}$$

with $H'_{\mu\nu}$ defined by lowering indices in (2.14) and

$$\boldsymbol{R}' = \boldsymbol{D}' \times \boldsymbol{B}' - \boldsymbol{E}' \times \boldsymbol{H}'/c^2, \tag{2.28}$$

when evaluated in the fluid rest frame (in which, e.g., $\delta\gamma = (\gamma^3/2c)\boldsymbol{v} \cdot \delta \boldsymbol{v} = 0$, since $\boldsymbol{v} = 0$ in the fluid rest frame). Hence, from (2.21) we find

$$\delta L_0 = \tfrac{1}{2} H'_{\mu\nu} \delta F'^{\mu\nu} - n'\theta' \delta s' - m_0 c^2 \delta n' - \left(\frac{P' + \varepsilon'}{n'} \right) \delta n' - \boldsymbol{R}' \cdot \delta \boldsymbol{v}. \tag{2.29}$$

Next, define the 4-vector with rest-frame components

$$R'_\mu = \left(\frac{P' + \varepsilon'}{c}, \boldsymbol{R}' \right) \tag{2.30}$$

and evaluate $\delta \Gamma^\mu$ from (2.23) in the rest frame

$$(\delta \Gamma')^\mu = (c\,\delta n', n'\,\delta \boldsymbol{v}). \tag{2.31}$$

Consequently, we have

$$\frac{R'_\mu}{n'} (\delta \Gamma')^\mu = \left(\frac{P' + \varepsilon'}{n'} \right) \delta n' + \boldsymbol{R}' \cdot \delta \boldsymbol{v}. \tag{2.32}$$

Then, from (2.25), (2.29), and (2.32) the following variational identity results, in mixed-frame notation:

$$\delta L_0 = \tfrac{1}{2} H'_{\mu\nu} \delta F'^{\mu\nu} - \frac{R'_\mu}{n'} (\delta \Gamma')^\mu + m_0 \bar{v}_\mu \delta \Gamma^\mu - n'\theta' \delta s'. \tag{2.33}$$

By virtue of its covariant form, (2.33) holds in any Lorentz frame. Thus, the primes can be dropped on Lorentz scalar density combinations in (2.33) to give

$$\delta L_0 = \tfrac{1}{2} H_{\mu\nu} \delta F^{\mu\nu} + \left(m_0 \bar{v}_\mu - \frac{R_\mu}{n'} \right) \delta \Gamma^\mu - n'\theta' \delta s. \tag{2.34}$$

The variables $\bar{v}_\mu$, R_μ, s, and n in (2.34) transform from the fluid rest frame (denoted by prime superscript) to the laboratory inertial frame (unadorned) via the following Lorentz transformation rules (see, e.g., Møller [18], eq. (IV.25′)):

$$\begin{aligned} &\bar{v}_\mu = \gamma(-c, \boldsymbol{v}) =: (\bar{v}_0, \bar{\boldsymbol{v}}), \\ &R_\mu = \gamma\left(\frac{P' + \varepsilon'}{c} - \frac{\boldsymbol{v} \cdot \boldsymbol{R}'_\parallel}{c},\ \gamma^{-1} \boldsymbol{R}'_\perp + \boldsymbol{R}'_\parallel - \boldsymbol{v}\left(\frac{P' + \varepsilon'}{c^2} \right) \right) =: (R_0, \boldsymbol{R}), \\ &s = s', \quad n = \gamma n', \end{aligned} \tag{2.35}$$

where parallel and perpendicular three-vector components are defined by

$$\boldsymbol{R}'_\parallel = \boldsymbol{v}(\boldsymbol{v} \cdot \boldsymbol{R}')/v^2, \quad \boldsymbol{R}'_\perp = \boldsymbol{R}' - \boldsymbol{R}'_\parallel. \tag{2.36}$$

Formulas (2.35) and (2.36) will be useful in section 3 in writing the SREMF equations in $3+1$ Hamiltonian form in the laboratory frame.

Hamilton's principle. Independent variations with respect to $\{\Gamma^\mu, \xi, \beta, \lambda_\Sigma, A_\mu, s', X^\Sigma\}$ in Hamilton's principle (2.16) using Lagrangian density (2.17) and variational formula (2.34) now produce the following dynamical equations:

$$\delta\Gamma^\mu \;:\; K_\mu := \left(m_0\bar{v}_\mu - \frac{R_\mu}{n'}\right) = -\xi_{,\mu} + \beta s'_{,\mu} + \lambda_\Sigma X^\Sigma_{,\mu}, \tag{2.37a}$$

$$\delta\xi \;:\; \Gamma^\mu_{,\mu} = 0, \tag{2.37b}$$

$$\delta\beta \;:\; \Gamma^\mu s'_{,\mu} = 0, \tag{2.37c}$$

$$\delta\lambda_\Sigma \;:\; \Gamma^\mu X^\Sigma_{,\mu} = 0, \tag{2.37d}$$

$$\delta A_\mu \;:\; H^{\mu\nu}_{,\nu} = 0, \tag{2.37e}$$

$$\delta s' \;:\; 0 = (\beta\Gamma^\mu)_{,\mu} - n'\theta' \; [\text{by (2.37b)}] = \Gamma^\mu\beta_{,\mu} - n'\theta', \tag{2.37f}$$

$$\delta X^\Sigma \;:\; 0 = (\lambda_\Sigma\Gamma^\mu)_{,\mu} \; [\text{by (2.37b)}] = \Gamma^\mu\lambda_{\Sigma,\mu}. \tag{2.37g}$$

The Lagrange multipliers ξ, β, λ_Σ, and X^Σ can be eliminated as in Penfield and Haus [1] by contracting Γ^ν with the 4-curl of (2.37a) to find

$$\Gamma^\nu(K_{\mu,\nu} - K_{\nu,\mu}) = n'\theta' s'_{,\mu}. \tag{2.38}$$

Penfield and Haus [1] observe that (2.38) is equivalent to

$$T_\mu{}^\nu{}_{,\nu} = 0, \tag{2.39}$$

with total energy-momentum tensor

$$T^{\mu\nu} = T^{\mu\nu}_{\mathrm{k}} + T^{\mu\nu}_{\mathrm{M}} + T^{\mu\nu}_{\mathrm{mat}}, \tag{2.40}$$

where the summands in (2.40) are defined by

$$T^{\mu\nu}_{\mathrm{k}} = m_0\bar{v}^\mu\Gamma^\nu, \tag{2.41a}$$

$$T^{\mu\nu}_{\mathrm{M}} = F^\mu_\kappa H^{\kappa\nu} - \tfrac{1}{2}(F_{\kappa\lambda}H^{\kappa\lambda})\delta^{\mu\nu}, \tag{2.41b}$$

$$T^{\mu\nu}_{\mathrm{mat}} = \delta^{\mu\nu}(P' + \bar{v}_\kappa T^{\kappa\lambda}_M\bar{v}_\lambda - \bar{v}_\kappa R^\kappa), \tag{2.41c}$$

and subscript k denotes kinetic, M denotes Minkowskii, and mat denotes material.

This completes our review of the Penfield–Haus variational principle. (See also Penfield and Haus [2] for further discussion.) Various components of the total energy-momentum tensor (2.40) will reappear in section 3, in the course of constructing the Hamiltonian formulation of the SREMF equations.

2.3. *Lie derivative property and Kelvin's theorem*

Eq. (2.38) has the interesting property of being expressible as the Lie derivative of the relativistic circulation one-form, $K_\mu\,\mathrm{d}x^\mu$, with respect to the 4-vector field, $\Gamma = \Gamma^\nu\partial_\nu$. Namely,

$$\mathscr{L}_\Gamma(K_\mu\,\mathrm{d}x^\mu) = \theta' n'\,\mathrm{d}_4 s' + \mathrm{d}_4(K_\nu\Gamma^\nu), \tag{2.42a}$$

where

$$K_\mu = m_0 \bar{v}_\mu - R_\mu / n', \tag{2.42b}$$

and d_4 is the exterior derivative in Minkowskii space (e.g., $\mathrm{d}_4 f = f_{,\mu}\, \mathrm{d}x^\mu$, for a Lorentz-scalar function f). See e.g., Schutz [20] for the standard properties used in Lie-derivative manipulations. Physically, eq. (2.42a, b) is the covariant form of Kelvin's theorem for relativistic electromagnetic fluids. Taking the Minkowskii exterior derivative d_4 of eq. (2.42a) gives the relativistic Helmholtz equation, cf. Lichnerowicz [17], eq. (21–5),

$$\mathscr{L}_\Gamma \mathrm{d}_4 (K_\mu\, \mathrm{d}x^\mu) = \mathrm{d}_4(\theta' n') \wedge \mathrm{d}_4 s', \tag{2.43}$$

where $\wedge$ denotes exterior product. Likewise, since by eq. (2.39),

$$\mathscr{L}_\Gamma s = \Gamma^\nu \partial_\nu s = 0, \tag{2.44}$$

we have (since d_4 commutes with $\mathscr{L}_\Gamma$)

$$\mathscr{L}_\Gamma\, \mathrm{d}_4 s = 0. \tag{2.45}$$

Consequently, there is another relativistic advection law, namely

$$\mathscr{L}_\Gamma \big(\mathrm{d}_4 (K_\mu\, \mathrm{d}x^\mu) \wedge \mathrm{d}_4 s\big) = 0, \tag{2.46}$$

by (2.43), (2.45), and the chain rule for Lie derivatives. The physical meaning of eqs. (2.43) and (2.46) will become clear when expressed in 3 + 1 form, in a moment.

In 3 + 1 form, the Lie-derivative property of eq. (2.38) as expressed in eq. (2.42a) persists and its interpretation as Kelvin's theorem is more obvious. Taking the ith component of (2.38), using $\bar{v}_\nu = \gamma(-c, \boldsymbol{v})$ from (2.35) in the laboratory frame, and defining the operator

$$\Gamma^\nu \partial_\nu = n' \bar{v}^\nu \partial_\nu = \gamma n' \frac{\mathrm{d}}{\mathrm{d}t}, \tag{2.47}$$

gives the three-dimensional equation equivalent to the spatial part of (2.38),

$$\frac{\mathrm{d}}{\mathrm{d}t}(m_0 \gamma v_i - R_i/n') = -\left(m_0 c^2 \gamma + \frac{P' + \varepsilon'}{n'}\right)_{,i} + v^j (m_0 \gamma v_j - R_j/n')_{,i} + \gamma^{-1} \theta' s'_{,i} \tag{2.48a}$$

$$= -\left(\frac{m_0 c^2}{\gamma} + \frac{P' + \varepsilon' + \boldsymbol{v} \cdot \boldsymbol{R}}{n'}\right)_{,i} - (m_0 \gamma v_j - R_j/n') v^j_{,i} + \gamma^{-1} \theta' s'_{,i}. \tag{2.48b}$$

Eq. (2.48b) may be rewritten in 3 + 1 Lie-derivative form, with d the *spatial* exterior derivative in three dimensions, as

$$(\partial_t + \mathscr{L}_v)(m_0 \gamma \boldsymbol{v} - \boldsymbol{R}/n') \cdot \mathrm{d}\boldsymbol{x} = -\mathrm{d}\left(\frac{m_0 c^2}{\gamma} + \frac{P' + \varepsilon' + \boldsymbol{v} \cdot \boldsymbol{R}}{n'}\right) + \gamma^{-1} \theta'\, \mathrm{d}s'. \tag{2.49}$$

Using the Lorentz formulas (2.35) now simplifies (2.49) in the laboratory frame to the expression

$$(\partial_t + \mathscr{L}_v)C = -\mathrm{d}\left(\frac{m_0 c^2 w}{\gamma}\right) + \gamma^{-1}\theta' \mathrm{d}s, \tag{2.50}$$

where: C is the total circulation one-form in three dimensions in the laboratory frame, namely,

$$C = \boldsymbol{C} \cdot \mathrm{d}\boldsymbol{x} := \left[m_0 \gamma w \boldsymbol{v} - \gamma\left(\boldsymbol{R}'_\perp + \gamma \boldsymbol{R}'_\parallel\right)/n\right] \cdot \mathrm{d}\boldsymbol{x}; \tag{2.51}$$

w is the relativistic specific enthalpy defined in (2.10); and the Lorentz transformation rules (2.26) and (2.35) give

$$\begin{aligned}\gamma \boldsymbol{R}'_\perp + \gamma^2 \boldsymbol{R}'_\parallel &= \gamma\left(\boldsymbol{D}' \times \boldsymbol{B}' - \boldsymbol{E}' \times \boldsymbol{H}'/c^2\right)_\perp + \gamma^2\left(\boldsymbol{D}' \times \boldsymbol{B}' - \boldsymbol{E}' \times \boldsymbol{H}'/c^2\right)_\parallel \\ &= \gamma^2\left[\left(\boldsymbol{D} \times \boldsymbol{B} - \boldsymbol{E} \times \boldsymbol{H}/c^2\right) - \boldsymbol{v} \times \left(\boldsymbol{B} \times \boldsymbol{H} + \boldsymbol{D} \times \boldsymbol{E}\right)/c^2\right].\end{aligned} \tag{2.52}$$

The relation (2.50) with circulation C defined in (2.51) is Kelvin's Theorem for SREMF in the laboratory frame. In particular for isentropic fluids, $\mathrm{d}s = 0$ and (2.50) expresses conservation in that case of circulation C in (2.51) integrated around closed paths moving with the fluid. The corresponding relation for isentropic fluids in Minkowskii space is eq. (2.42) with $\mathrm{d}_4 s'$ set equal to zero.

Two immediate consequences of Kelvin's theorem appear when (2.50) is combined with the corresponding Lie-derivative expressions for the dynamics of n and s, namely

$$(\partial_t + \mathscr{L}_v)s = 0, \tag{2.53}$$

$$(\partial_t + \mathscr{L}_v)(n\, \mathrm{d}^3 x) = 0, \tag{2.54}$$

where $\mathrm{d}^3 x$ is the three-dimensional volume element in the laboratory frame. First, one finds the advection rule

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = 0, \tag{2.55}$$

for the SREMF "potential vorticity" Ω defined by

$$\Omega = n^{-1} \nabla s \cdot \operatorname{curl} \boldsymbol{C}, \tag{2.56}$$

where the 3-vector $\boldsymbol{C}$ appears in (2.51)

$$\boldsymbol{C} = m_0 \gamma w \boldsymbol{v} - \gamma\left(\boldsymbol{R}'_\perp + \gamma \boldsymbol{R}'_\parallel\right)/n. \tag{2.57}$$

The proof of (2.55) uses the standard properties of exterior derivatives and Lie derivatives (see, e.g., Schutz [20]). In particular, d^2 vanishes and d commutes with $\mathscr{L}_v$, so that one finds (with $\wedge$ denoting exterior product, cf. eq. (2.46) in four-dimensional notation)

$$(\partial_t + \mathscr{L}_v)(\mathrm{d}C \wedge \mathrm{d}s) = 0. \tag{2.58}$$

This follows by direct computation from (2.50), (2.51), and (2.53), using the chain rule for Lie derivatives. In coordinate notation (2.58) becomes the local conservation law

$$\partial_t(\nabla s \cdot \operatorname{curl} \boldsymbol{C}) + \operatorname{div}[\boldsymbol{v}(\nabla s \cdot \operatorname{curl} \boldsymbol{C})] = 0. \tag{2.59}$$

Then (2.55) follows from (2.59) by the continuity equation (2.37b), or equivalently (2.54).

Remark. The potential vorticity advection equation (2.55) along with (2.53) and (2.54) imply global conservation of the laboratory-frame quantity

$$C_F = \int_D \mathrm{d}^3x\, nF(\Omega, s), \tag{2.60}$$

for an *arbitrary* function F, provided $\boldsymbol{v}$ is tangential to the boundary ∂D of the spatial region of integration D. That is,

$$\partial_t C_F = 0, \quad \text{provided } \boldsymbol{v}\cdot\hat{\boldsymbol{n}}|_{\partial D} = 0, \tag{2.61}$$

for impermeable boundaries. At this stage, the proof that C_F in (2.60) is conserved follows merely as a computational fact. A Lie algebraic reason for the conservation of C_F will be given in the Hamiltonian context described in the next section.

The second consequence of (2.50) is conservation for appropriate boundary and initial conditions of the SREMF helicity, Λ, defined by

$$\Lambda = \int_D \mathrm{d}^3x\, \boldsymbol{C}\cdot \operatorname{curl} \boldsymbol{C} = \int_D C \wedge \mathrm{d}C. \tag{2.62}$$

In geometrical language, the proof of helicity conservation for SREMF proceeds by the following computation:

$$\begin{aligned}
(\partial_t + \mathscr{L}_v) C \wedge \mathrm{d}C \,[\text{by chain rule}] &= C \wedge (\partial_t + \mathscr{L}_v)\mathrm{d}C + [(\partial_t + \mathscr{L}_v)C] \wedge \mathrm{d}C \\
[\text{by (2.50)}] &= C \wedge \mathrm{d}(\gamma^{-1}\theta'\mathrm{d}s) + \left[\mathrm{d}\left(-\frac{c^2 w}{\gamma}\right) + \gamma^{-1}\theta'\mathrm{d}s\right] \wedge \mathrm{d}C \\
&= -\mathrm{d}\left[\frac{c^2 w}{\gamma}\mathrm{d}C - C \wedge \gamma^{-1}\theta'\mathrm{d}s\right] + 2\gamma^{-1}\theta'\mathrm{d}s \wedge \mathrm{d}C.
\end{aligned} \tag{2.63}$$

In coordinates, (2.63) becomes, with $\lambda = \boldsymbol{C}\cdot \operatorname{curl} \boldsymbol{C}$,

$$\partial_t \lambda = -\operatorname{div}\left[\lambda \boldsymbol{v} + \frac{c^2 w}{\gamma}\operatorname{curl} \boldsymbol{C} - \gamma^{-1}\theta' \boldsymbol{C} \times \nabla s\right] + 2\gamma^{-1}\theta'\Omega \tag{2.64}$$

$$=: -\operatorname{div} \boldsymbol{F}_{(\lambda)} + 2\gamma^{-1}\theta'\Omega, \tag{2.65}$$

with helicity flux $\boldsymbol{F}_{(\lambda)}$ given by the square-bracketed term in (2.64). Consequently, with surface element $\mathrm{d}S$

and outward normal unit vector $\hat{\boldsymbol{n}}$ on the boundary ∂D, eq. (2.65) gives

$$\partial_t \Lambda = \partial_t \int_D \lambda \, \mathrm{d}^3x = -\oint_{\partial D} \boldsymbol{F}_{(\lambda)} \cdot \hat{\boldsymbol{n}} \, \mathrm{d}S + \int_D 2\gamma^{-1}\theta' \Omega \, \mathrm{d}^3x. \tag{2.66}$$

Consequently, the SREMF helicity in (2.62) will be preserved for boundary conditions such that the surface integral in (2.66) vanishes, and initial conditions of vanishing Ω. Note that by (2.55) if Ω is initially zero throughout the domain of flow, it will remain zero. Physically, the helicity Λ measures the number of linkages of the lines of total electromagnetic fluid vorticity, curl $\boldsymbol{C}$. (See Moffatt [21] for discussions of helicity in other fluid theories, e.g., in nonrelativistic magnetohydrodynamics, where it plays a role in the dynamo effect.)

Section summary. In this section we have recounted a version of the Penfield–Haus variational principle for SREMF, modified slightly to take A_μ as a basic variable. Furthermore, we have studied the differential-geometric significance of Kelvin's theorem, both covariantly and after the $3+1$ split that moves us into the laboratory frame. Conservation laws for potential vorticity and helicity have emerged as natural consequences of Kelvin's theorem in this differential-geometric setting. The presence of the Lie derivatives in Kelvin's theorem (2.50) hints strongly that Lie algebras are present in the underlying structure of SREMF. These Lie algebras will appear explicitly in the Hamiltonian formulation of the SREMF equations, given in the next section.

The rest of this paper is devoted to: (1) determining the Hamiltonian formulation of SREMF and discussing Lyapunov stability of its equilibrium solutions; (2) finding the relation of this Hamiltonian formulation to that of special relativistic magnetohydrodynamics (SRMHD), and taking the nonrelativistic (NR) limits of both formulations. In the NR limit, the SREMF Hamiltonian formulation returns to that for NREMF given in Holm [12], and the SRMHD Hamiltonian formulation returns to the Hamiltonian formulation for NRMHD given in Morrison and Greene [22], which is derived and interpreted mathematically in Holm and Kupershmidt [4].

3. Hamiltonian formulation of SREMF

From the action principle (2.16) with Lagrangian density (2.17) one passes to the Hamiltonian formulation in the laboratory frame by Legendre transforming with respect to the canonically conjugate momenta, defined by

$$P_{(I)} = \frac{\delta L}{\delta q^{(I)}_{,0}}, \quad I \in \{\xi, s, X^\Sigma, A_i\}. \tag{3.1}$$

Thus, we find the canonical momentum formulas

$$\begin{aligned} P_{(\xi)} &= \frac{\delta L}{\delta \xi_{,0}} = nc, \quad P_{(s)} = \frac{\delta L}{\delta s_{,0}} = -\beta nc, \\ P_{(X^\Sigma)} &= \frac{\delta L}{\delta X^\Sigma_{,0}} = -\lambda_\Sigma nc, \quad P_{(A_i)} = \frac{\delta L}{\delta A_{i,0}} = H^{i0} = -D^i. \end{aligned} \tag{3.2}$$

The corresponding Hamiltonian density is calculated as follows:

$$\mathscr{H} = \sum_I P_{(I)} q^{(I)}_{,0} - L \tag{3.3a}$$

$$[\text{by (2.37a)}] = nm_0c^2\gamma w - P' - D^iA_{i,0} - \boldsymbol{E}'\cdot\boldsymbol{D}' - \boldsymbol{v}\cdot\left(\gamma\boldsymbol{R}'_\perp + \gamma^2\boldsymbol{R}'_\parallel\right) \tag{3.3b}$$

$$[\text{by (2.52)}] = nm_0c^2\gamma w - P' - A_0\operatorname{div}\boldsymbol{D} + \boldsymbol{E}\cdot\boldsymbol{D} - \boldsymbol{E}'\cdot\boldsymbol{D}' - \gamma^2\boldsymbol{v}\cdot\left(\boldsymbol{D}\times\boldsymbol{B} - \boldsymbol{E}\times\boldsymbol{H}/c^2\right). \tag{3.3c}$$

Note that $\mathscr{H} = T^{00}$, by expressions (2.40) and (2.41) for the energy-momentum tensor $T^{\mu\nu}$ evaluated in the laboratory frame. Note also that the Clebsch variables no longer appear in (3.3c), so the Hamiltonian density is expressed entirely in terms of physical variables.

In the conventional Hamiltonian formalism, the starting system (2.1) and (2.2) can now be shown to be expressible as a Hamiltonian system,

$$\partial_t F(P, q) = \{H, F\}_c, \tag{3.4}$$

with Hamiltonian $H = \int \mathrm{d}^3x\,\mathscr{H}$ and canonical (symplectic) Poisson bracket

$$\{H, F\}_c = \sum_I \int \mathrm{d}^3x \left[\frac{\delta F}{\delta q^{(I)}}\frac{\delta H}{\delta P_{(I)}} - \frac{\delta H}{\delta q^{(I)}}\frac{\delta F}{\delta P_{(I)}}\right], \tag{3.5}$$

for $I \in \{\xi, s, X^\Sigma, A_i\}$. The success of such an approach is guaranteed, since at this stage we are merely Legendre-transforming an action principle that we know produces the correct equations from the previous section.

There is, however, a more expeditious procedure; namely, to construct a *noncanonical* Hamiltonian formalism directly in the space of physical variables $\{M_i = c^{-1}T_i^{\,0}, n, s, A_i, -D^i\}$, by finding a canonical map (i.e., a map that preserves Poisson brackets) from the space of canonical variables $(P_{(I)}, q^{(I)})$ in (3.2) to the smaller space of physical variables. Such a map already appears in the Clebsch variational equations of the previous section. We define the "circulation vector" from (2.51) and the spatial part of the Clebsch equation (2.37a),

$$\boldsymbol{C} = -\nabla\xi + \beta\nabla s + \lambda_\Sigma \nabla X^\Sigma; \tag{3.6}$$

whence, we find for the 3-vector momentum density in terms of the canonical variables [see (2.51) and (2.52)]

$$M_i := c^{-1}T_i^{\,0} = nC_i = m_0\gamma wnv_i - \gamma\left(\boldsymbol{R}'_\perp + \gamma\boldsymbol{R}'_\parallel\right)_i = -\sum_I P_{(I)} q^{(I)}_{,i}, \tag{3.7a}$$

where, now,

$$I \in \{\xi, s, X^\Sigma\},$$

not including the vector potential, A_i. We also have the remainder of the Clebsch map,

$$n = c^{-1}P_{(\xi)}, \quad s = q^{(s)}, \quad A_i = q^{(A_i)}, \quad D^i = -P_{(A_i)}. \tag{3.7b, c, d, e}$$

In particular, the electromagnetic piece of the Poisson bracket will be unchanged under this map.

The Clebsch map (3.7a)–(3.7e) will induce a correct Poisson bracket in the space of physical variables $\{M_i, n, s, A_i, -D^i\}$, provided the following formula holds (see, e.g., Holm and Kupershmidt [4] and Holm [16]):

$$\phi(\mathbf{B}) = \frac{\mathrm{D}Z}{\mathrm{D}Y} \cdot \mathbf{b} \cdot \left(\frac{\mathrm{D}Z}{\mathrm{D}Y}\right)^{\dagger}, \tag{3.8}$$

where : ϕ is the map from the canonical space with coordinates Y, into the physical space with coordinates Z; $\mathbf{b}$ is the canonical matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$; $\mathbf{B}$ is the Hamiltonian matrix in the physical space; $\phi(\mathbf{B})$ is computed by applying the map ϕ to *each* matrix element of $\mathbf{B}$; $\mathrm{D}Z/\mathrm{D}Y$ is the Frechet derivative of the variables Z with respect to the variables Y; and the symbol † denotes adjoint with respect to the volume measure $\mathrm{d}^3x = \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$.

In the present case, the Hamiltonian matrix in the space of physical variables that results from this procedure is (see Holm and Kupershmidt [4] and Holm, Kupershmidt, and Levermore [23])

$$-\mathbf{B} = \begin{array}{c|ccccc|} & M_j & n & s & A_j & D_j \\ \hline M_i & M_j\partial_i + \partial_j M_i & n\partial_i & -s_{,i} & 0 & 0 \\ n & \partial_j n & 0 & 0 & 0 & 0 \\ s & s_{,j} & 0 & 0 & 0 & 0 \\ A_i & 0 & 0 & 0 & 0 & -\delta_i^j \\ D_i & 0 & 0 & 0 & \delta_j^i & 0 \end{array}, \tag{3.9}$$

where $s_{,i} := (\partial s/\partial x^i)$ and the differential operator ∂_j acts on whatever stands to its right. The Poisson bracket corresponding to the Hamiltonian matrix (3.9) is given by

$$\{H, F\} = -\int \mathrm{d}^3x \, \frac{\delta F}{\delta Z} \cdot \mathbf{B} \cdot \frac{\delta H}{\delta Z}$$

$$= -\int \mathrm{d}^3x \left\{ \frac{\delta F}{\delta M_i} \left[\left(M_j\partial_i + \partial_j M_i\right) \frac{\delta H}{\delta M_j} + n\partial_i \frac{\delta H}{\delta n} - s_{,i} \frac{\delta H}{\delta s} \right] + \left[\frac{\delta F}{\delta n} \partial_j n + \frac{\delta F}{\delta s} s_{,j} \right] \frac{\delta H}{\delta M_j} \right\} \tag{3.10a}$$

$$-\int \mathrm{d}^3x \left[\frac{\delta F}{\delta A_i} \frac{\delta H}{\delta D^i} - \frac{\delta H}{\delta A_i} \frac{\delta F}{\delta D^i} \right], \tag{3.10b}$$

where, again, ∂_i operates on terms to its right.

Remark on the Lie algebra associated to Poisson bracket (3.10*a*). The Poisson bracket (3.10a, b) is bilinear, skew adjoint, and satisfies the Jacobi identity. (The first two properties are obvious.) We verify the Jacobi identity by observing that the Poisson bracket (3.10a) is the natural Poisson bracket on the dual to the semidirect product Lie algebra $V \circledS [\Lambda^0 \oplus \Lambda^3]$ (see Holm and Kupershmidt [4]), where $V = V(\mathbb{R}^3)$ represents vector fields on $\mathbb{R}^3$ (X_j denotes elements of V) and $\Lambda^k = \Lambda^k(\mathbb{R}^3)$ denotes k-forms on $\mathbb{R}^3$. V acts on itself by commutation of vector fields (denoted by [,]) and acts upon Λ^k by Lie derivation, denoted, e.g., $X(\xi^{(k)})$ for $\xi^{(k)} \in \Lambda^k$. The symbol $\circledS$ denotes semidirect product, and $\oplus$ the direct sum. The Lie algebraic commutator corresponding to the Poisson bracket (3.10a) is

$$\left[\left(X;\xi^{(0)}\oplus\xi^{(3)}\right),\left(\bar{X};\bar{\xi}^{(0)}\oplus\bar{\xi}^{(3)}\right)\right]=\left(\left[X,\bar{X}\right];\left(X(\bar{\xi}^{(0)})-\bar{X}(\xi^{(0)})\right)\oplus\left(X(\bar{\xi}^{(3)})-\bar{X}(\xi^{(3)})\right)\right). \tag{3.11}$$

Dual coordinates are: M_i dual to $X_i \in V$; n, to $\xi^{(0)} \in \Lambda^0$; and s, to $\xi^{(3)} \in \Lambda^3$.

Poisson brackets such as (3.10a) that are associated to the dual of a Lie algebra are called "Lie–Poisson" brackets in the nomenclature of Marsden et al. [6] and Weinstein [24]. See also Kupershmidt [25] for further discussion of such Poisson brackets.

SREMF equations. Using eqs. (3.7a) and (2.52) for $\boldsymbol{M}$, namely,

$$\boldsymbol{M}=m_0\gamma w n\boldsymbol{v}-\tilde{\boldsymbol{R}}, \tag{3.12a}$$

$$\tilde{\boldsymbol{R}}:=\gamma^2\left[\left(\boldsymbol{D}\times\boldsymbol{B}-\boldsymbol{E}\times\boldsymbol{H}/c^2\right)-\boldsymbol{v}\times\left(\boldsymbol{B}\times\boldsymbol{H}+\boldsymbol{D}\times\boldsymbol{E}\right)/c^2\right], \tag{3.12b}$$

the Hamiltonian from eq. (3.3c) may be rewritten simply as

$$H=\int \mathrm{d}^3x\left[\boldsymbol{M}\cdot\boldsymbol{v}+m_0c^2n'+\varepsilon'+\boldsymbol{E}\cdot\boldsymbol{D}-\boldsymbol{E}'\cdot\boldsymbol{D}'\right]. \tag{3.13}$$

The variational derivatives of this Hamiltonian are determined from the formula

$$\begin{aligned}\delta H=\int \mathrm{d}^3x\,&\left[\boldsymbol{v}\cdot\delta\boldsymbol{M}+\left(m_0c^2w\gamma^{-1}\right)\delta n+\left(\theta' n\gamma^{-1}\right)\delta s+\operatorname{curl}\boldsymbol{H}\cdot\delta\boldsymbol{A}+\boldsymbol{E}\cdot\delta\boldsymbol{D}\right.\\&\left.+\left(\boldsymbol{M}-m_0\gamma w n\boldsymbol{v}+\tilde{\boldsymbol{R}}\right)\cdot\delta\boldsymbol{v}\right],\end{aligned} \tag{3.14}$$

which follows from (2.7), (2.10), (2.27), and (2.34).

Substituting the variational derivatives from (3.14) into the Lie–Poisson bracket (3.10a, b) leads immediately to

$$\partial_t n=\{H,n\}=-(nv^j)_{,j}, \tag{3.15a}$$

$$\partial_t s=\{H,s\}=-s_{,j}v^j, \tag{3.15b}$$

$$\partial_t M_i=\{H,M_i\}=-M_jv^j_{,i}-\left(M_iv^j\right)_{,j}-n\left(m_0c^2w/\gamma\right)_{,i}+\gamma^{-1}n\theta' s_{,i}, \tag{3.15c}$$

$$\partial_t\boldsymbol{A}=\{H,\boldsymbol{A}\}=-\boldsymbol{E}, \tag{3.15d}$$

$$\partial_t\boldsymbol{D}=\{H,\boldsymbol{D}\}=\operatorname{curl}\boldsymbol{H}. \tag{3.15e}$$

Eqs. (3.15a, b) reproduce the subsidiary equations (2.1b, c) of continuity and adiabaticity, respectively. Eqs. (3.15d, e) are Maxwell's equations, which carry over unchanged from the action principle, since the Poisson map (3.7a–e) does not interfere with the electromagnetic field part of the canonical bracket. Eq. (3.15c) can be re-expressed in terms of the circulation vector C_i to recover (2.50) in coordinate form. Indeed, substituting $M_i = nC_i$ from (3.7a) into (3.15c) and using (3.15a) gives

$$\partial_tC_i+v^jC_{i,j}+C_jv^j_{,i}=-\left(m_0c^2w\gamma^{-1}\right)_{,i}+\gamma^{-1}\theta' s_{,i}, \tag{3.16}$$

which is (2.50) in coordinate notation.

Remark on Casimirs. In (2.60) we derived the conservation law for

$$C_F = \int_D \mathrm{d}^3x\, nF(\Omega, s), \quad \Omega = n^{-1}\nabla s \cdot \operatorname{curl} \boldsymbol{C},$$
$$\boldsymbol{C} = \boldsymbol{M}/n = m_0\gamma w \boldsymbol{v} - \gamma^2 n^{-1}\left[\boldsymbol{D}\times\boldsymbol{B} - \boldsymbol{E}\times\boldsymbol{H}/c^2 - \boldsymbol{v}\times(\boldsymbol{B}\times\boldsymbol{H} + \boldsymbol{D}\times\boldsymbol{E})/c^2\right], \tag{3.17}$$

for an arbitrary function F of potential vorticity Ω defined in (2.56) and of specific entropy s. The conserved functional C_F in (3.17) is a "Casimir" in the sense that

$$\{C_F, G\} = 0, \quad \forall G(\{M_i, n, s, \boldsymbol{A}, \boldsymbol{D}\}). \tag{3.18}$$

That is, C_F lies in the kernel of the Lie–Poisson bracket (3.10a) and is, thus, conserved *independently of the choice of Hamiltonian* (or perturbations to a given Hamiltonian) in the space of physical variables.

The conservation of C_F in (3.17) can be understood as resulting from the "gauge symmetry" of the Hamiltonian H in (3.13), under canonical transformations generated by C_F *in the space of canonical variables* $(P_{(I)}, q^{(I)})$; $I \in \{\xi, s, X^{\Sigma}, A_i\}$. Such gauge transformations in the Clebsch variables are "trivial" in the sense of preserving the values of the physical variables at each point in space. The infinitesimal canonical transformations corresponding to C_F are expressible in terms of the original symplectic Poisson bracket $\{,\}_\mathrm{c}$ as

$$\delta P_{(I)} = \{C_F, P_{(I)}\}_\mathrm{c} = -\frac{\delta C_F}{\delta q^{(I)}},$$
$$\delta q^{(I)} = \{C_F, q^{(I)}\}_\mathrm{c} = \frac{\delta C_F}{\delta P_{(I)}}. \tag{3.19}$$

Since the Hamiltonian (3.13) is expressed solely in terms of the physical variables, by (3.18) the Clebsch gauge transformations (3.19) are Hamiltonian symmetries, and their generators C_F are conserved.

Recently, Casimirs such as (3.17) have been used to determine Lyapunov stability criteria for ideal fluid and plasma equilibria in a variety of situations, see, e.g., Abarbanel et al. [26, 27], Arnold [28, 29], Holm, Marsden, Ratiu, and Weinstein [30, 8], and Holm and Kupershmidt [9]. In this regard, we note that critical points of the sum $H + C_F$, i.e., of the energy plus Casimir, will be equilibrium states of the SREMF equations in the laboratory frame; since the critical-point condition $\delta(H + C_F) = 0$ implies that the dynamical variables have no time dependence (see eq. (4.11)). Associating equilibrium states with critical points of conserved functionals is of interest because this step is the starting point of the Lyapunov stability method, discussed in the next section.

Section summary. In this section, we have produced the Hamiltonian formulation of SREMF dynamics using the Lie–Poisson bracket (3.10a, b) expressed directly in terms of the physical variables. The bracket (3.10a) is associated to the dual of the semidirect-product Lie algebra with commutator (3.11). This formulation, while not canonical, does reveal an infinite number of conservation laws of physical significance, namely the C_F in eq. (3.17). These conservation laws (due, in the present setting, to Clebsch gauge symmetries of the Penfield–Haus variational principle) are in involution, i.e., are mutually commuting, but of course are not functionally independent (so complete integrability is not an issue here). The initial hint that Lie algebraic structure exists in these equations came from the interpretation of Kelvin's theorem (2.50) as a differential-geometric statement.

Next, we study Lyapunov stability conditions for EMF equilibria. These conditions will emerge naturally on considering critical points of $H + C_F$, where C_F given in (3.17) is in the kernel of the Lie–Poisson bracket (3.10a).

4. Lyapunov stability for SREMF equilibria

4.1. *Equilibrium states*

Lyapunov stability of equilibria for the SREMF equations may be studied using the Casimir constants of motion C_F lying the kernel of the Lie–Poisson bracket (3.10). These Casimirs in three dimensions are given by (3.17). To find equilibrium-state relations for the three-dimensional SREMF equations (3.15a–e), we first rewrite (3.16) as

$$\partial_t \boldsymbol{C} = \boldsymbol{v} \times \operatorname{curl} \boldsymbol{C} - \nabla\left(c^2 w \gamma^{-1} + \boldsymbol{v} \cdot \boldsymbol{C}\right) + \gamma^{-1} \theta' \, \nabla s. \tag{4.1}$$

Taking the curl of (4.1) and using eqs. (3.15a–b) gives

$$\partial_t \Omega = -\boldsymbol{v} \cdot \nabla \Omega, \tag{4.2}$$

with Ω defined in (3.17). Thus, by (3.15a, b, d, e), (4.1), and (4.2), the equilibrium states $(n_e, s_e, \boldsymbol{v}_e, \boldsymbol{D}_e, \boldsymbol{B}_e)$ satisfy the following relations:

$$\operatorname{curl} \boldsymbol{E}_e = 0, \tag{4.3a}$$

$$\operatorname{curl} \boldsymbol{H}_e = 0, \tag{4.3b}$$

$$\operatorname{div}(n_e \boldsymbol{v}_e) = 0, \tag{4.3c}$$

$$\boldsymbol{v}_e \times \operatorname{curl} \boldsymbol{C}_e = \nabla\left(c^2 w_e \gamma_e^{-1} + \boldsymbol{v}_e \cdot \boldsymbol{C}_e\right) - \gamma_e^{-1} \theta_e' \, \nabla s_e, \tag{4.3d}$$

$$\boldsymbol{v}_e \cdot \nabla s_e = 0, \tag{4.3e}$$

$$\boldsymbol{v}_e \cdot \nabla \Omega_e = 0, \tag{4.3f}$$

where $\boldsymbol{C}_e$ and Ω_e are given by evaluating the definitions in (3.17) at equilibrium. Also included among the equilibrium relations are the nondynamical conditions

$$\operatorname{div} \boldsymbol{D}_e = 0 \quad \text{and} \quad \operatorname{div} \boldsymbol{B}_e = 0. \tag{4.4}$$

In (4.3a) and (4.3b) the quantities $\boldsymbol{E}_e$ and $\boldsymbol{H}_e$ are determined by Lorentz-transforming the thermodynamic derivatives in (2.7) at equilibrium to the laboratory frame. Scalar multiplication of (4.3d) by $\boldsymbol{v}_e$ and use of (4.3e) imply [together with (4.3e) and (4.3f)] the existence of three orthogonality relations with $\boldsymbol{v}_e$ at equilibrium,

$$\boldsymbol{v}_e \cdot \nabla\left(c^2 w_e \gamma_e^{-1} + \boldsymbol{v}_e \cdot \boldsymbol{C}_e\right) = 0, \quad \boldsymbol{v}_e \cdot \nabla s_e = 0, \quad \boldsymbol{v}_e \cdot \nabla \Omega_e = 0. \tag{4.5}$$

In turn, these three orthogonality relations imply a functional relationship among the three quantities s_e, Ω_e, and $(c^2 w_e \gamma^{-1} + \boldsymbol{v}_e \cdot \boldsymbol{C}_e)$, provided the equilibrium state is *nondegenerate*, i.e., provided $\nabla s_e \times \nabla \Omega_e \neq 0$

and $n_e \Omega_e v_e \neq 0$. This implied relationship is *assumed* to be expressible as a Bernoulli law, namely, as

$$c^2 w_e \gamma_e^{-1} + v_e \cdot C_e = K(s_e, \Omega_e), \tag{4.6}$$

for a function K, called the Bernoulli function. The equilibrium relation (4.3d) and the Bernoulli relation (4.6) then yield

$$v_e \times \operatorname{curl} C_e = \nabla K(s_e, \Omega_e) - \gamma_e^{-1} \theta_e' \, \nabla s_e. \tag{4.7}$$

Vector multiplication of this equation (4.7) by ∇s_e and use of (4.3e–f) determines, for $\Omega_e \neq 0$, that

$$n_e v_e = \Omega_e^{-1} K_\Omega \, \nabla s_e \times \nabla \Omega_e, \tag{4.8}$$

where $K_\Omega := \partial K(s_e, \Omega_e)/\partial \Omega_e$. Note that the divergence of (4.8) vanishes, as required by the equilibrium relation (4.3c). Similarly, vector multiplication of (4.7) by $\nabla \Omega_e$ and use of (4.8) gives the equilibrium relation

$$\frac{\nabla \Omega_e \cdot \operatorname{curl} C_e}{\nabla s_e \cdot \operatorname{curl} C_e} = \frac{\gamma_e^{-1} \theta_e' - K_s}{K_\Omega}, \tag{4.9}$$

where $K_s = \partial K(s_e, \Omega_e)/\partial s_e$. Finally, the equilibrium relations (4.3a–b) are satisfied when E_e and H_e are expressible as gradients of scalar functions.

Relations (4.6), (4.8), and (4.9) will be useful in characterizing equilibrium states of SREMF as critical points of the following functional, defined in the domain of flow D by

$$\begin{aligned} H_C &= H + C_F + \lambda \int_D n\Omega \, \mathrm{d}^3x + \int_D [\operatorname{div}(\mu \boldsymbol{D}) + \operatorname{div}(\nu \boldsymbol{B})] \, \mathrm{d}^3x \\ &= \int_D \mathrm{d}^3x \, [\boldsymbol{M} \cdot \boldsymbol{v} + m_0 c^2 n' + \varepsilon' + \boldsymbol{E} \cdot \boldsymbol{D} - \boldsymbol{E}' \cdot \boldsymbol{D}'] + \int_D \mathrm{d}^3x \, [nF(\Omega, s) + \lambda n\Omega + \operatorname{div}(\mu \boldsymbol{D} + \nu \boldsymbol{B})], \end{aligned} \tag{4.10}$$

where $\lambda = \text{const}$, and μ and ν are functions of $\boldsymbol{x}$, all to be determined. The λ-term is separated out from F for later convenience, and the μ and ν terms in the integrand are added to impose the nondynamical constraints $\operatorname{div} \boldsymbol{D} = 0$ and $\operatorname{div} \boldsymbol{B} = 0$; although these latter terms identically vanish for the imposed boundary conditions that $\boldsymbol{D}$ and $\boldsymbol{B}$ are tangential on the boundary. It follows from the Hamiltonian equations (3.15) and (3.18) that

$$\partial_t G = \{H + C_F, G\} = \{H_C, G\}. \tag{4.11}$$

Hence, critical points of H_C are equilibrium states of the motion equations (3.15a–e). The converse statement, namely that nondegenerate equilibrium flows of (3.15a–e) satisfying (4.6), (4.8), and (4.9) are critical points of H_C requires separate analysis, which we now provide.

The first variation of H_C in (4.10) yields (after integration by parts)

$$\delta H_C = \int_D \mathrm{d}^3x \,\{ (\boldsymbol{v} + n^{-1}\nabla F_\Omega \times \nabla s)\cdot\delta\boldsymbol{M} + (m_0c^2w\gamma^{-1} + F - \Omega F_\Omega - n^{-2}\boldsymbol{M}\cdot\nabla F_\Omega\times\nabla s)\,\delta n$$
$$+ [\theta' n\gamma^{-1} + nF_s - \nabla F_\Omega\cdot\mathrm{curl}(\boldsymbol{M}n^{-1})]\,\delta s + (\boldsymbol{E} + \nabla\mu)\cdot\delta\boldsymbol{D} + (\boldsymbol{H}+\nabla\nu)\cdot\delta\boldsymbol{B}\}$$
$$+ \oint_{\partial D} (F_\Omega + \lambda)[\delta s\,\mathrm{curl}(\boldsymbol{M}/n) - \nabla s\times(n^{-1}\delta\boldsymbol{M} - n^{-2}\boldsymbol{M}\delta n)]\cdot\hat{\boldsymbol{n}}\,\mathrm{d}S, \tag{4.12}$$

where $\hat{\boldsymbol{n}}\,\mathrm{d}S$ is the oriented surface element on the boundary, ∂D. When evaluated at equilibrium, the coefficients of the boundary integral will vanish upon choosing

$$F_\Omega(\Omega_\mathrm{e}, s_\mathrm{e})|_{\partial D} + \lambda = 0, \tag{4.13}$$

which is possible for boundary condition $\boldsymbol{v}_\mathrm{e}\cdot\hat{\boldsymbol{n}}|_{\partial D} = 0$; since then $F_\Omega|_{\partial D} = \mathrm{const.}$ The coefficients in (4.12) of the independent variations $(\delta\boldsymbol{M}, \delta n, \delta s, \delta\boldsymbol{B}, \delta\boldsymbol{D})$ will each vanish in the interior of domain D for equilibrium states, provided the following conditions hold:

$$\boldsymbol{v}_\mathrm{e} + n_\mathrm{e}^{-1}\nabla F_\Omega\times\nabla s_\mathrm{e} = 0, \tag{4.14a}$$
$$m_0c^2w_\mathrm{e}\gamma_\mathrm{e}^{-1} + F_\mathrm{e} - \Omega_\mathrm{e}F_\Omega - n_\mathrm{e}^{-2}\boldsymbol{M}_\mathrm{e}\cdot\nabla F_\Omega\times\nabla s_\mathrm{e} = 0, \tag{4.14b}$$
$$\theta_\mathrm{e}' n_\mathrm{e}\gamma_\mathrm{e}^{-1} + n_\mathrm{e}F_s - \nabla F_\Omega\cdot\mathrm{curl}(n_\mathrm{e}^{-1}\boldsymbol{M}_\mathrm{e}) = 0, \tag{4.14c}$$
$$\boldsymbol{E}_\mathrm{e} + \nabla\mu = 0, \tag{4.14d}$$
$$\boldsymbol{H}_\mathrm{e} + \nabla\nu = 0. \tag{4.14e}$$

We now show that each of these conditions holds by virtue of the relations (4.3)–(4.9) for nondegenerate SREMF equilibria. Condition (4.14a) can be rewritten as

$$n_\mathrm{e}\boldsymbol{v}_\mathrm{e} = F_{\Omega\Omega}\,\nabla s_\mathrm{e}\times\nabla\Omega_\mathrm{e}, \tag{4.15}$$

which is equivalent to the equilibrium state relation (4.8), provided

$$F_{\Omega\Omega} = \Omega_\mathrm{e}^{-1}K_\Omega. \tag{4.16}$$

Substitution of (4.14a) into (4.14b) gives

$$m_0c^2w_\mathrm{e}\gamma_\mathrm{e}^{-1} + n_\mathrm{e}^{-1}\boldsymbol{v}_\mathrm{e}\cdot\boldsymbol{M}_\mathrm{e} + F_\mathrm{e} - \Omega_\mathrm{e}F_\Omega = 0, \tag{4.17}$$

or equivalently by the Bernoulli relation (4.6) for equilibrium states,

$$K + F_\mathrm{e} - \Omega_\mathrm{e}F_\Omega = 0, \tag{4.18}$$

which is a first integral of (4.16) with respect to Ω_e. Both relations (4.16) and (4.18) will be satisfied, so that both the δn and $\delta\boldsymbol{M}$ coefficients will vanish simultaneously, provided F is determined from K by solving (4.18), namely,

$$F(\beta, \sigma) = \beta\left(\int^\beta \frac{K(\sigma, z)}{z^2}\,\mathrm{d}z + \psi(\sigma)\right), \tag{4.19}$$

where ψ is an arbitrary function. This is the fundamental relation between the Bernoulli function K and the Casimir function F for equilibrium states.

Substituting (4.18) into (4.14c) now gives the equilibrium state relation (4.9). Finally, critical point relations (4.14d) and (4.14e) imply the equilibrium state relations (4.3a) and (4.3b), respectively. This proves the following proposition:

Proposition 4.1. For smooth solutions satisfying $\boldsymbol{v}\cdot\hat{\boldsymbol{n}}=0$, $\boldsymbol{B}\cdot\hat{\boldsymbol{n}}=0$, and $\boldsymbol{D}\cdot\hat{\boldsymbol{n}}=0$ on the boundary, a nondegenerate equilibrium state $(\boldsymbol{v}_e, n_e, s_e, \boldsymbol{D}_e, \boldsymbol{B}_e)$ of the ideal SREMF equations is a critical point of

$$H_C = H + C_F + \lambda \int_D n\Omega\, d^3x + \int_D \operatorname{div}(\mu\boldsymbol{D} + \nu\boldsymbol{B})]\, d^3x, \tag{4.20}$$

provided relation (4.18) is satisfied between the Bernoulli function K in (4.6) and the Casimir function F in (3.17), and eq. (4.13) is satisfied on the boundary. Here the functions $\mu(\boldsymbol{x})$ and $\nu(\boldsymbol{x})$ ensure $\operatorname{div}\boldsymbol{D}=0$ and $\operatorname{div}\boldsymbol{B}=0$, respectively, and are related to the equilibrium fields by (4.14d, e); λ is a constant; C_F is defined in (3.17); and H is given in (3.13).

Remark. For flow on a planar surface of constant specific entropy s, the gradient ∇s becomes a vector normal to the plane, which for motion in the xy plane gives $\Omega = n^{-1}\hat{z}\cdot\operatorname{curl}\boldsymbol{C}$, up to a constant factor. For this case, in the first variation formula (4.12) the variation δs is absent and ∇s is replaced by the unit vector $\hat{z}$, normal to the plane.

4.2. *Linear Lyapunov stability of equilibrium states*

The linearization of the Hamiltonian system (3.15a–e) around an equilibrium state $(n_e, s_e, \boldsymbol{v}_e, \boldsymbol{D}_e, \boldsymbol{B}_e)$ that is a critical point of its Hamiltonian-plus-Casimir, H_C, is again a Hamiltonian system, whose conserved Hamiltonian function is given by one-half the second variation $\delta^2 H_C$, see Holm et al. [8]. Consequently, if $\delta^2 H_C$ is definite in sign as a quadratic form, then it provides a conserved norm that measures deviations from equilibrium of an initial disturbance under the linearized dynamics. Therefore, the conditions on the equilibrium flow for $\delta^2 H_C$ to be definite are sufficient conditions for linear Lyapunov stability. That is, a flow that starts near an equilibrium solution satisfying these conditions will remain (under the linearized dynamics) within a neighborhood of this solution determined from the norm given by $\delta^2 H_C$. This is the essence of the Lyapunov stability method we use. This method is based on two main ideas: (1) characterization of equilibrium flows as critical points of certain functionals composed of constants of motion; and (2) linearized preservation in time of the second variations of these functionals considered as norms for linear Lyapunov stability. In most fluid cases, such stability may be strengthened to *nonlinear* Lyapunov stability by using an additional convexity argument, see Holm et al. [8].

Using these two main ideas, we now seek the conditions on the nondegenerate equilibrium SREMF flows for the second variation $\delta^2 H_C$ to be definite and, thus, provide a stability norm.

In the basis $(\delta\boldsymbol{M}, \delta n, \delta s, \delta\boldsymbol{D}, \delta\boldsymbol{B})$ we have

$$\delta^2 H_C = \int_D d^3x \left[F_{\Omega\Omega}(\delta\Omega)^2 + Q(\delta\boldsymbol{M}, \delta n, \delta s, \delta\boldsymbol{D}, \delta\boldsymbol{B}) \right], \tag{4.21}$$

where Q is a rather complicated quadratic form whose leading coefficient in $|\delta\boldsymbol{M}|^2$ is $m_0\gamma_e w_e n_e$, which is positive; and $\delta\Omega$ is given by

$$\delta\Omega = D\Omega\,(\boldsymbol{M}_e, n_e, s_e, \boldsymbol{D}_e, \boldsymbol{B}_e)\cdot(\delta\boldsymbol{M}, \delta n, \delta s, \delta\boldsymbol{D}, \delta\boldsymbol{B}). \tag{4.22}$$

In particular, if Q in (4.21) is positive, a sufficient condition for stability (i.e., for positive definiteness of $\delta^2 H_C$) is that the equilibrium flow should satisfy

$$F_{\Omega\Omega}(\Omega_e, s_e) = [\text{by (4.15)}]\,\frac{n_e \boldsymbol{v}_e \cdot \nabla s_e \times \nabla\Omega_e}{|\nabla s_e \times \nabla\Omega_e|^2} > 0. \tag{4.23}$$

Remark. For isentropic flow in the xy plane, the sufficient condition for stability (4.23) becomes

$$\frac{n_e \boldsymbol{v}_e \cdot \hat{\boldsymbol{z}} \times \nabla\Omega_e}{|\nabla\Omega_e|^2} > 0, \quad \text{with } \Omega_e = n_e^{-1}\hat{\boldsymbol{z}}\cdot\operatorname{curl}\left(n_e^{-1}\boldsymbol{M}_e\right). \tag{4.24}$$

As a corollary, we find for *planar* shear flows, namely flows satisfying

$$\boldsymbol{M}_e = M_e(y)\hat{\boldsymbol{x}}, \quad n_e = n_e(y),$$

that a *necessary* condition for *instability* is that the quantity $[n_e^{-1}\mathrm{d}(n_e^{-1}M_e)/\mathrm{d}y]$ have an extremal point as a function of the transverse coordinate y; then, the quantity on the left side of (4.24) would violate the inequality by passing through zero. This corollary generalizes Rayleigh's [1880] inflection point criterion necessary for instability of a shear flow of a planar ideal incompressible fluid to the case of SREMF.

5. Other canonical maps and reduction to SRMHD

5.1. *The entangling map*

Section 3 shows that SREMF equations possess a Poisson bracket (3.10a, b) dual to a Lie algebra, expressible symbolically as

$$L_1 = \left\{ V \circledS [\Lambda^0 \oplus \Lambda^3] \right\} \oplus \omega(\alpha, \phi), \tag{5.1}$$

with dual coordinates: $\boldsymbol{M}$ dual to V; n, to Λ^0; s, to Λ^3; $\boldsymbol{D}$, to $\alpha \in \Lambda^1$; and $\boldsymbol{A}$ to $\phi \in \Lambda^2$; while the canonical $\boldsymbol{D} - \boldsymbol{A}$ part of the bracket is dual to the two-cocycle $\omega(\alpha, \phi)$, which is symplectic.

Holm [1986] shows that the *nonrelativistic* EMF equations also possess another Poisson bracket, dual to a *different* Lie algebra, expressible as

$$L_2 = V \circledS [\Lambda^0 \oplus \Lambda^3 \oplus \Lambda^1 \oplus \Lambda^2] \oplus \omega(\alpha, \phi), \tag{5.2}$$

with the *same* dual coordinates and two-cocycle, *except* for a redefinition of momentum density, called $\boldsymbol{N}$ instead of $\boldsymbol{M}$ and defined below in (5.3a).

Since the nonrelativistic limit should be regular and structure preserving (Holm and Kupershmidt [31]), there must be a canonical map between the Poisson brackets corresponding to L_1 and L_2. This map, called the "entangling" map, is related to the difference in definitions of $\boldsymbol{M}$ and $\boldsymbol{N}$ in the Poisson brackets dual to L_1 or L_2, respectively. This map "entangles" the variables in the Poisson bracket in the sense that in going from L_1 to L_2 the variables $\boldsymbol{D}$ and $\boldsymbol{A}$ (dual to the elements of the two-cocycle in L_2) no longer Poisson-commute with $\boldsymbol{N}$ (dual to vector fields V). The inverse of this map (the "*un*tangling" map) is due to Alan Weinstein [32] and Holm and Kupershmidt [36]. The map also can be derived from a constrained action principle for MHD, leading to a Clebsch representation analogous to (1.1) but with $\boldsymbol{M}$ replaced by $\boldsymbol{N}$, as given below in (5.3). A useful theorem regarding untangling the canonically-conjugate variables from the others in a Lie–Poisson bracket appears in Krishnaprasad and Marsden [33].

In passing from the Poisson bracket corresponding to L_1 to that for L_2, one uses the appropriate modification of the matrix relation (3.8) for the entangling map,

$$\begin{aligned}&\boldsymbol{N}=\boldsymbol{M}+\boldsymbol{D}\times\operatorname{curl}\boldsymbol{A}-\boldsymbol{A}\operatorname{div}\boldsymbol{D},\\&n=n,\quad s=s,\quad \boldsymbol{A}=\boldsymbol{A},\quad \boldsymbol{D}=\boldsymbol{D},\end{aligned}\tag{5.3}$$

resulting in the following Poisson bracket dual to L_2 in (5.2), namely:

$$\begin{aligned}\{H,F\}=-\int \mathrm{d}^3x\bigg\{&\frac{\delta F}{\delta N_i}\bigg[\left(N_j\partial_i+\partial_jN_i\right)\frac{\delta H}{\delta N_j}+n\partial_i\frac{\delta H}{\delta n}-s_{,i}\frac{\delta H}{\delta s}\\&+\left(D_j\partial_i-\partial_kD^k\delta_{ij}\right)\frac{\delta H}{\delta D_j}+\left(\partial_jA_i-A_{j,i}\right)\frac{\delta H}{\delta A_j}\bigg]+\bigg[\frac{\delta F}{\delta n}\partial_jn+\frac{\delta F}{\delta s}s_{,j}\bigg]\frac{\delta H}{\delta N_j}\\&+\frac{\delta F}{\delta D_i}\left(\partial_jD_i-D^k\partial_k\delta_{ij}\right)\frac{\delta H}{\delta N_j}+\frac{\delta F}{\delta A_i}\left(A_j\partial_i+A_{i,j}\right)\frac{\delta H}{\delta N_j}\bigg\}\end{aligned}\tag{5.4a}$$

$$-\int \mathrm{d}^3x\left[\frac{\delta F}{\delta A_i}\delta_{ij}\frac{\delta H}{\delta D_j}-\frac{\delta F}{\delta D_i}\delta_{ij}\frac{\delta H}{\delta A_j}\right].\tag{5.4b}$$

The variational derivatives of the EMF Hamiltonian H (3.13) in the new basis (5.3) are given by [changing variables in (3.14) and integrating by parts]

$$\delta H=\int \mathrm{d}^3x\left[v^j\,\delta N_j+\left(m_0c^2w/\gamma\right)\delta n+\left(\theta'n/\gamma\right)\delta s+\left(\operatorname{curl}\boldsymbol{H}^*\right)\cdot\delta\boldsymbol{A}+\boldsymbol{E}^*\cdot\delta\boldsymbol{D}\right],\tag{5.5}$$

where, in the notation of Pauli [34],

$$\boldsymbol{E}^*=\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B},\tag{5.6a}$$

$$\boldsymbol{H}^*=\boldsymbol{H}-\boldsymbol{v}\times\boldsymbol{D}.\tag{5.6b}$$

The quantities $\boldsymbol{E}^*$ and $\boldsymbol{H}^*$ are, respectively, the electric field intensity and magnetic field intensity as measured in the fluid frame. This is apparent from the 4-vector relations for the "electric vector"

$$\begin{aligned}e^\nu&=c^{-1}\bar{v}_\mu F^{\mu\nu}=\gamma\left(\boldsymbol{v}\cdot\boldsymbol{E}/c,\,\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B}\right)\\&=\gamma\left(\boldsymbol{v}\cdot\boldsymbol{E}^*/c,\,\boldsymbol{E}^*\right)\end{aligned}\tag{5.7}$$

and "magnetic vector"

$$\begin{aligned}h^\nu&=\tilde{H}^{\nu\mu}\bar{v}_\mu=\gamma\left(\boldsymbol{v}\cdot\boldsymbol{H}/c,\,\boldsymbol{H}-\boldsymbol{v}\times\boldsymbol{D}\right)\\&=\gamma\left(\boldsymbol{v}\cdot\boldsymbol{H}^*/c,\,\boldsymbol{H}^*\right),\end{aligned}\tag{5.8}$$

where $F^{\mu\nu}$ is given by (2.12) with $\boldsymbol{E} \to -\boldsymbol{E}$ and $\tilde{H}^{\nu\mu}$ is the dual tensor to $H^{\mu\nu}$ in (2.14). Namely,

$$\tilde{H}^{\nu\mu} = \begin{vmatrix} 0 & H_1/c & H_2/c & H_3/c \\ & 0 & -D_3 & D_2 \\ & & 0 & -D_1 \\ & & & 0 \end{vmatrix}. \tag{5.9}$$

Substitution of the variational derivatives of H obtained from (5.5) into the Poisson bracket (5.4a, b) readily yields the dynamical equations of SREMF. In particular, we verify (2.1b, c) by inspection just as in (3.14a, b). Namely,

$$\partial_t n = \{H, n\} = -(nv^j)_{,j}, \tag{5.10}$$

$$\partial_t s = \{H, s\} = -s_{,j} v^j. \tag{5.11}$$

Next, Maxwell's equations follow, by

$$\begin{aligned} \partial_t D_i = \{H, D_i\} &= -\left(D_i v^j\right)_{,j} + D^k v_{i,k} + \varepsilon_{ijk} H^*_{k,j} \\ [\text{by (4.6b)}] &= [\operatorname{curl}(\boldsymbol{v} \times \boldsymbol{D})]_i - v_i \operatorname{div} \boldsymbol{D} + (\operatorname{curl} \boldsymbol{H})_i - [\operatorname{curl}(\boldsymbol{v} \times \boldsymbol{D})]_i \\ [\text{by (2.15a)}] &= (\operatorname{curl} \boldsymbol{H})_i \end{aligned} \tag{5.12}$$

and

$$\begin{aligned} \partial_t A_i = \{H, A_i\} &= -A_j v^j_{,i} - A_{i,j} v^j - E_i^* \\ [\text{by (4.6a)}] &= -\left(A_j v^j\right)_{,i} + v^j\left(A_{j,i} - A_{i,j}\right) - \boldsymbol{E}_i - (\boldsymbol{v} \times \boldsymbol{B})_i \\ &= -\left(A_j v^j\right)_{,i} - E_i. \end{aligned} \tag{5.13}$$

Thus, the $\boldsymbol{A}$-equation changes by a gradient from (3.14d); but its curl, the flux equation (2.15d) remains the same when the Poisson bracket (3.10a, b) undergoes the entangling map (5.3) to become (5.4a, b).

5.2. *Reduction of SREMF to SRMHD*

Before writing the dynamics of N_i, we map the Poisson bracket (5.4a, b) from $\boldsymbol{A}$ to $\boldsymbol{B} = \operatorname{curl} \boldsymbol{A}$, leaving the other variables unchanged. Again using the Hamiltonian matrix relation (3.8), we find yet another Poisson bracket (see Holm [12])

$$\begin{aligned} \{H, F\} = -\int \mathrm{d}^3x \Bigg\{ \frac{\delta F}{\delta N_i} \Bigg[&\left(N_j \partial_i + \partial_j N_i\right) \frac{\delta H}{\delta N_j} + n \partial_i \frac{\delta H}{\delta n} - s_{,i} \frac{\delta H}{\delta s} \\ &+ \left(D_j \partial_i - \partial_k D^k \delta_{ij}\right) \frac{\delta H}{\delta D_j} + \left(B_j \partial_i - \partial_k B^k \delta_{ij}\right) \frac{\delta H}{\delta B_j} \Bigg] + \left[\frac{\delta F}{\delta n} \partial_j n + \frac{\delta F}{\delta s} s_{,j}\right] \frac{\delta H}{\delta N_j} \\ &+ \frac{\delta F}{\delta D_i}\left(\partial_j D_i - D^k \partial_k \delta_{ij}\right) \frac{\delta H}{\delta N_j} + \frac{\delta F}{\delta B_i}\left(\partial_j B_i - B^k \partial_k \delta_{ij}\right) \frac{\delta H}{\delta N_j} \Bigg\} \end{aligned} \tag{5.14a}$$

$$- \int \mathrm{d}^3x \left[\frac{\delta F}{\delta D_i} \varepsilon_{ijk} \partial_k \frac{\delta H}{\delta B_j} - \frac{\delta F}{\delta B_i} \varepsilon_{ijk} \partial_k \frac{\delta H}{\delta D_j} \right], \tag{5.14b}$$

where δ_{ij} is the Kronecker delta and ε_{ijk} is the totally antisymmetric tensor density in three dimensions.

The Lie-algebraic interpretation of Poisson bracket (5.14) is given in Holm [12]. Since this map and the entangling map are both canonical, all three of the Poisson brackets (3.10), (5.4), and (5.14) share the same Casimirs.

The variational derivatives of H now are given by [cf. eq. (5.5)]

$$\delta H = \int \mathrm{d}^3x \left[v^j \delta N_j + \left(m_0 c^2 w/\gamma \right) \delta n + \left(\theta' n/\gamma \right) \delta s + \boldsymbol{H}^* \cdot \delta \boldsymbol{B} + \boldsymbol{E}^* \cdot \delta \boldsymbol{D} \right]. \tag{5.15}$$

Thus, we find by a calculation analogous to (4.12)

$$\begin{aligned} \partial_t B_i = \{ H, B_i \} &= -\left(B_i v^j \right)_{,j} + B^k v_{i,k} - (\operatorname{curl} \boldsymbol{E}^*)_i \\ [\text{by (4.6a)}] &= [\operatorname{curl}(\boldsymbol{v} \times \boldsymbol{B})]_i - v_i \operatorname{div} \boldsymbol{B} - (\operatorname{curl} \boldsymbol{E})_i - [\operatorname{curl}(\boldsymbol{v} \times \boldsymbol{B})]_i \\ [\text{by (2.15c)}] &= -(\operatorname{curl} \boldsymbol{E})_i, \end{aligned} \tag{5.16}$$

so the flux equation (2.15d) reappears from the Poisson bracket (5.14). The other equations (5.10)–(5.12) are unchanged by the map to the Poisson bracket for SREMF (5.14) in the $\boldsymbol{B}$-representation.

Finally, the dynamics of N_i is given in conservative form as

$$\begin{aligned} \partial_t N_i = \{ H, N_i \} &= -\partial_k \Bigg[\left(n \frac{\delta H}{\delta n} + s \frac{\delta H}{\delta s} + N_j \frac{\delta H}{\delta N_j} + D_j \frac{\delta H}{\delta D_j} + B_j \frac{\delta H}{\delta B_j} - \mathscr{H} \right) \delta_i^k \\ &\qquad + N_i \frac{\delta H}{\delta N_k} - D^k \frac{\delta H}{\delta D^i} - B^k \frac{\delta H}{\delta B^i} \Bigg] \\ [\text{by (4.15)}] &= -\partial_k \Big[\left(m_0 c^2 w n \gamma^{-1} + s \theta' n \gamma^{-1} + N_j v^j + D_j E^{*j} + B_j H^{*j} - \mathscr{H} \right) \delta_i^k \\ &\qquad + N_i v^k - D^k E_i^* - H_i^* B^k \Big]. \end{aligned} \tag{5.17a}$$

Or, equivalently in geometrical form as

$$(\partial_t + \mathscr{L}_v)(n^{-1} N_i \, \mathrm{d}x^i) = -\mathrm{d} \frac{\delta H}{\delta n} + n^{-1} \frac{\delta H}{\delta s} \mathrm{d}s + \mathscr{L}_{n^{-1}\boldsymbol{D}}(\boldsymbol{D} \cdot \mathrm{d}\boldsymbol{x}) + \mathscr{L}_{n^{-1}\boldsymbol{B}}(\boldsymbol{B} \cdot \mathrm{d}\boldsymbol{x}). \tag{5.17b}$$

In the nonrelativistic limit, eq. (5.17b) reduces to eq. (19b) of Holm [12], see also Calkin [13] for the case of nonrelativistic polarized fluids without magnetization.

When the electric displacement vector $\boldsymbol{D}$ is absent, the momentum density $\boldsymbol{N}$ in (5.3a) becomes [cf. (2.51) and (2.52)]

$$\boldsymbol{N} = m_0 \gamma w n \boldsymbol{v} + \gamma^2 \boldsymbol{E} \times \boldsymbol{H}/c^2 + \gamma^2 \boldsymbol{v} \times (\boldsymbol{B} \times \boldsymbol{H}). \tag{5.18}$$

If also $\boldsymbol{B} = \mu_0 \boldsymbol{H}$ and $\boldsymbol{E} = -\boldsymbol{v} \times \boldsymbol{B}$, then the last term in (5.18) vanishes, and $\boldsymbol{N}$ becomes the momentum density for SRMHD (eq. (17) of Holm and Kupershmidt [35]). The variational derivatives of H in (5.15) and, hence, the equations of motion also become those for SRMHD. (See Holm and Kupershmidt [35].) Finally, the further limit to nonrelativistic MHD proceeds regularly and uniformly as c^{-2} tends to zero, thereby recovering the noncanonical Hamiltonian density formulation for NRMHD originally due to Morrison and Greene [22].

Remark. Lyapunov stability of equilibria for SRMHD may be studied using the constants of motion for the kernel of the Lie–Poisson bracket (5.14a) with $\boldsymbol{D}$ absent. (The mathematical interpretation of the resulting bracket is given in Holm and Kupershmidt [35].) The equilibria corresponding to $\delta(H + C_F) = 0$ in this case, however, are stationary in the laboratory frame, so relativistic effects are nonexistent for these equilibria.

Section summary. This section shows how to pass via canonical maps from one Hamiltonian formulation for SREMF to another. In the second formulation, SRMHD is contained as a special case: namely, $\boldsymbol{D}$ absent, $\boldsymbol{B} = \mu_0 \boldsymbol{H}$, $\boldsymbol{E} = -\boldsymbol{v} \times \boldsymbol{B}$. Nonrelativistic EMF and MHD are then obtained as regular limits of the special relativistic theories when c^{-2} tends to zero.

6. Conclusions

This paper has treated the Hamiltonian structure underlying the theory of ideal special-relativistic neutral electromagnetic fluids with induction. This Hamiltonian structure has been identified as a Lie–Poisson structure, dual to a semidirect-product Lie algebra. Casimirs – constants of motion lying in the kernel of this Lie algebra – have been identified and used to classify equilibrium solutions and study their stability properties. In particular, an extremum criterion necessary for instability of those equilibria has been found for planar SREMF shear flows. Approximate descriptions of SREMF (particularly SRMHD) have been constructed via the Lie algebraic nature of the Hamiltonian formulation of SREMF and the nonrelativistic limits of these approximations have been determined.

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